Negativity and Concurrence for two qutrits

Suranjana Rai[†] and Jagdish R. Luthra[*]
(†) Raitech, Tuscaloosa, AL 35405
(∗) Departamento de Físíca, Universidad de los Andes, A.A. 4976 Bogotá, Colombia

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Abstract

Two measures of entanglement, negativity and concurrence are studied for two qutrits. An operator origin of negativity is presented and an analytic formula connecting the two measures is derived.

Entanglement is well understood for two qubits. A good measure of entanglement is the concurrence defined by Wootters [1]. It is related to the entanglement of formation (EOF) which, for pure states, is the entropy of the reduced density matrix. A great deal of work is going on to understand both multipartite entanglement and entanglement of higher dimensional systems. But a good measure or set of measures to completely specify general qudit entanglement has not yet been found. In this paper, we explore the connection between different measures of entanglement for pure states of two qutrits, in particular, the relation between negativity and concurrence.

Generalizing concurrence to higher-dimensional bipartite states is not a mere extension of the two-qubit concurrence. A generalized concurrence known as the I-concurrence has been obtained for qudits[2]. But I-concurrence is not a fully satisfactory measure of entanglement since it is not a monotonic function of the EOF. It does indeed reduce to the EOF for some special cases, but not in general.

We look at another measure of entanglement, the negativity defined by Vidal and Werner [3]. Negativity is an entanglement monotone so it does not change under local operations and classical communications (LOCC). Vidal defines negativity of a bipartite system described by the density matrix ρ, as $N(\rho)$

$$N(\rho) \quad = \quad \frac{||\rho^{T_A}||_1 - 1}{2} \tag{1}$$

where ρ^{T_A} is the partial transpose with respect to system A, $||...||_1$ denotes the trace norm. Negativity is a quantitative measure of the partial positive transpose (PPT), the Peres criteria [4]. It measures how negative the eigenvalues of

[*]e-mail: jluthra@uniandes.edu.co

the density matrix are after the partial transpose is taken. It is the absolute value of the sum of the negative eigenvalues of the partially transposed density matrix. According to the Peres criterion of separability, density matrices with NPT are entangled [4]. The Peres criterion is necessary and sufficient for qubit-qubit and qubit-qutrit systems. For systems of higher dimensions, it is necessary but not sufficient, since, there are states with PPT which are entangled. These states are said to have bound entanglement since entanglement can not be distilled from these states.

The significance of negativity can be seen from the quantity logarithmic negativity, which Vidal and Werner go on to define as

$$E_N(\rho) \quad = \quad log_2||\rho_A^T||_1. \tag{2}$$

E_N is an upper bound on the entanglement of distillation(EOD), which for pure states reduces to the von Neumann entropy. But $E_N(\rho) \geq E(\rho)$, so negativity gives a larger value than the entropy and the equality holds for pure maximally entangled states.

Lee et. al., generalize the negativity to higher dimensions [5]

$$N(\rho) \quad = \quad \frac{||\rho^{T_A}||_1 - 1}{d-1} \tag{3}$$

where d is smaller of the dimension of the bipartite system. We use this negativity and it ranges from zero to one.

A general bipartite pure state for qutrits in the Schmidt form can be written as

$$|\psi\rangle \quad = \quad \sum_{i=1}^{d} k_i|ii\rangle \tag{4}$$

with normalization condition

$$k_1^2 + k_2^2 + k_3^2 \quad = \quad 1 \tag{5}$$

where k_1, k_2 and k_3 are the Schmidt coefficients, which are non-negative real numbers.

The negativity N of this state $|\psi\rangle$ is given in terms of Schmidt coefficients as

$$N(\rho) \quad = \quad \sum_{i<j} k_i k_j \tag{6}$$

For the two qutrit pure state, the negativity has the explicit form

$$N(\rho) \quad = \quad (k_1k_2 + k_2k_3 + k_3k_1) \tag{7}$$

We also show that the negativity can be obtained by the action of a ladder operator[6], $X = X_1 \otimes X_2$, where X_1 acts on the first qutrit and X_2 acts on the second qutrit. The operator X is defined as $X|11\rangle = |22\rangle, X|22\rangle = |33\rangle$ and $X|33\rangle = |11\rangle$.

In general

$$X|i,i\rangle = |i+1, i+1\rangle, \text{ mod(d)} \tag{8}$$

The operator X transforms the state $|\psi\rangle$ into a shifted state by the ladder action. Just as Wootters concurrence is defined using the Pauli spin matrix σ_y as a spin flip operator, we define the action of the ladder operator X, on a state and take the inner product with the original state. This is similar in spirit to the origin of concurrence for qubits. But it turns out to be not the concurrence as in the qubit case but the Vidal negativity. The operator X is analogous to the flip operator σ_y. It is easy to see that negativity can be obtained as the expectation value of the X operator in the state $|\psi\rangle$.

$$N(\rho) = \langle X\rangle = (k_1k_2 + k_2k_3 + k_3k_1) \tag{9}$$

This expression is the same as Eq.(7). This equation also holds in general for qudits

$$N(\rho) = \langle X\rangle = \sum_{i<j} k_ik_j \tag{10}$$

We now connect the concurrence to the negativity. Cereceda [7] has obtained a result for the concurrence [7] C of two qutrits on the lines of the I-concurrence [2].

$$C(|\psi\rangle \quad = \quad \sqrt{3(k_1^2k_2^2 + k_2^2k_3^2 + k_3^2k_1^2)} \tag{11}$$

The negativity and concurrence provide two important measures of entanglement. We are interested in exploring the relation between negativity N and concurrence C and the conditions under which the two are equal. Perhaps this connection might be useful for finding the various quantities that are necessary for a complete description of the entanglement. There exists a simple algebraic relation between N and C for two qutrits

$$N^2 \quad = \quad \frac{C^2}{3} \pm 2(k_1k_2k_3)\sqrt{1+2N} \tag{12}$$

Concurrence for two qubits is a monotonic function of the EOF, so it is a good measure of entanglement. However, concurrence in three dimensions is no longer a monotonic function of the EOF [7]. For qubits, concurrence uses conjugation. The conjugation is done by the Pauli operator σ_y. In higher dimensions, conjugation is not so simple. There is no one conjugated state. However, there is the inverted state which is the sum of all the states available to the system. A pure state after inversion goes in general to a mixed state. The

concurrence generated in this way has been called I-concurrence [2], as distinct from the Θ-concurrence [8], which is based on antilinear conjugation. For qutrits and higher dimensional systems, antilinear conjugation fails as transformation matrices are singular.

Both concurrence and negativity arise as a sum of entanglement between pairs of levels. Concurrence is the root mean square of pair-wise products of the Schmidt coefficients while negativity is the sum of pair-wise products of the coefficients. The difference contains a part which occurs with the product of the three Schmidt coefficients, $k_1k_2k_3$. This product vanishes when three-level entanglement is absent,i.e., when one of the Schmidt coefficient is zero, in which case the N and C differ from each other by just a scaling factor and now concurrence becomes a monotonic function of the EOF. That this difference is in terms of the explicit three-way entanglement of the two qutrits is possibly of some significance.

The negativity is upper bounded by the Cereceda concurrence. It is interesting to note that $C \geq N$. The maximum value of C and N are one for maximally entangled states.

Fu et. al. [9], have studied the violation of Clauser-Horne-Shimony-Holt (CHSH) for two qutrits. The degree of entanglement P_E they obtain is related to negativity as $P_E = N$. Cereceda has compared his concurrence C with P_E for states of two qutrits. We connect N with C for states with $k_1 = k_2 = \sqrt{x/2}$ and $k_3 = \sqrt{(1-x)}$. This is shown in Fig.1.

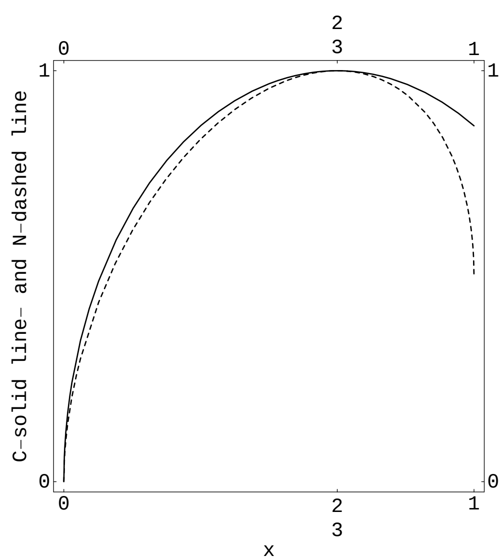

Figure 1: Concurrence C and negativity N for two qutrits in pure state. Concurrence C is shown by the solid upper line, and negativity N by the lower dashed line, against the parameter x, with $k_1 = k_2 = \sqrt{x/2}$ and $k_3 = \sqrt{(1-x)}$. See also Figs. 2 and 3 for a spherical parametrization of C and N, respectively.

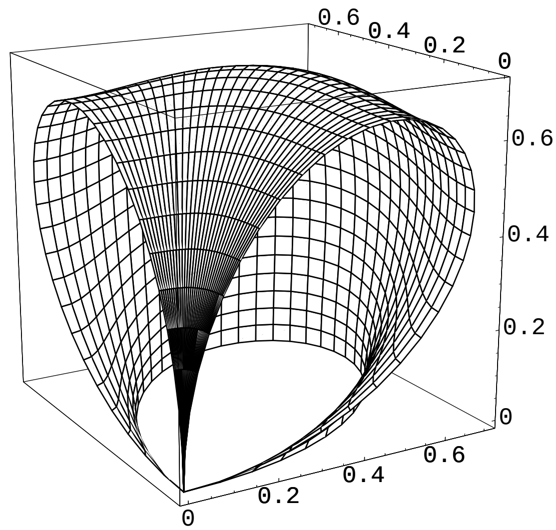

Figure 2: Concurrence C of two qutrits in a pure state, in spherical coordinates $k_1 = \sin\theta\cos\phi$, $k_2 = \sin\theta\sin\phi$, and $k_3 = \cos\theta$, for $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{2}$.

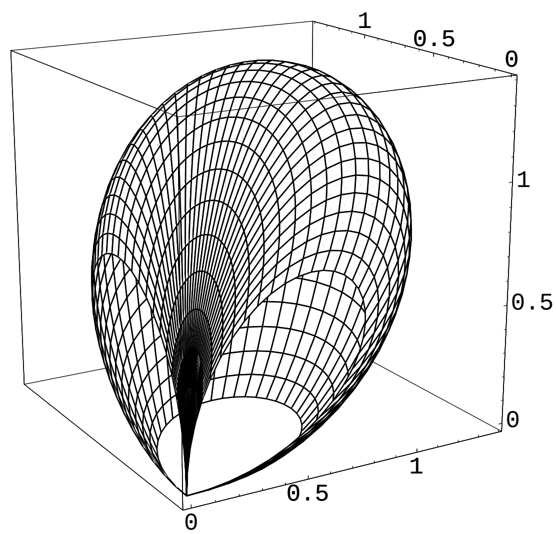

Figure 3: Negativity N of two qutrits in a pure state, in spherical coordinates $k_1 = \sin\theta\cos\phi$, $k_2 = \sin\theta\sin\phi$, and $k_3 = \cos\theta$, for $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{2}$.

When one of the $k's$ is zero, the two qutrit state reduces to a two-qubit state.

In this case, even though C goes from zero to $\sqrt{3}/2$, the N goes from zero to $1/2$.

In Fig. 2 and 3. we show the three-dimensional plots of N and C using a general three parameter two qutrit state. The maximum values of N and C are along the body diagonals of the cubes. The variation of C is flatter while the N has a more well defined lobe around its maximum direction.

In conclusion, N and C provide two useful quantities for describing the entanglement property and are found to be related. In higher dimensions, such a connection is more complicated but may be fruitful to study.

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